Correlation functions on the border lines of transient chaos

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Chaotic properties of a repeller strongly influence the transient properties of a system close to it, in particular the correlations in the transient regime. In this paper results are presented for repellers of one-dimensional maps having a fixed point whose Lyapunov exponent agrees with the escape rate from the repeller: It is proven that the corresponding natural measure of the repeller is a δ function at the origin. Eigenfunctions of the Frobenius-Perron operator are computed. The correlation function is calculated near the situation of permanent chaos and anomalous decay of the correlations is found. Scaling properties are given on the route from a weak repeller to a nonrepeller. The analytic results are supported by numerical calculations. [S1063-651X(96)07806-3]

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I. INTRODUCTION

It is well known that *in the long time limit* an attractor reflects the properties of all systems in its basin of attraction. This holds true for the ‘‘thermodynamic’’ properties and for the correlations observed in these systems. On the other hand a repeller [1,2] cannot influence the asymptotic properties of trajectories. Even when starting close to the repeller the trajectories will move away from it. Nevertheless the properties of repellers are important during a *transient time*: A system coming close to a repeller will remain there for some time (the transient time) and during this time the repeller will imprint its properties on the behavior of the system. The transient time depends on how closely the system approaches the repeller (if it is accidentally *on* the repeller the transient time is of course infinite but this case is not usual) and on the *escape rate* [3] of the repeller. So for a weak repeller (i.e., one with a small escape rate) the transient time is long and the correlations found during this time are determined intrinsically by the repeller. It is the aim of the present paper to compute these correlations for a simple one-dimensional (1D) discrete map and to demonstrate their universal properties.

To define a suitable correlation function we argue as follows: Imagine a weak repeller with an invariant measure ρ_r just before the transition to a nonrepeller state with a completely different invariant measure $\rho_{\rm nr}$ — an example for this transition will be given in the present paper. In this situation the measure $\rho_{\rm nr}$ will already be felt by the transients but will not influence the trajectories *on* the repeller. This emphasizes the importance of taking into account a neighborhood $\mathcal{U}$ of the repeller and thus we need a measure of the neighborhood $\mathcal{U}$. To become more specific we will restrict ourselves to discrete maps f for which a Frobenius-Perron operator $\mathcal{L}$ is defined. Then we can find this measure by the following procedure: We start with an arbitrary density $\rho_{\rm init}$ in $\mathcal{U}$. The change of $\rho_{\rm init}$ after each step of iteration is induced by the discrete map and given by $\mathcal{L}$. Repeated application of $\mathcal{L}$ will lead to a decrease of the density and to a change of its structure at the same time. Finally the density *normalized* in $\mathcal{U}$ is expected to converge to P, the density of the *conditionally invariant measure* [4], whereas the density itself is decreasing after each iteration by a certain rate, the *decay* rate:

$$\mathcal{L}P=\lambda_c P(\ \ln\lambda_c \text{ is the decay rate}). \tag{1}$$

Therefore, we use the conditionally invariant measure P when calculating averages:

$$\langle c_1(f^m)c_2\rangle=\frac{\int_{\mathcal{U}_m}c_1(f^m(x))c_2(x)P(x)dx}{\int_{\mathcal{U}_m}P(x)dx},$$

with

$$\mathcal{U}_m=f^{-m}(U)\cap U$$

Exploiting the properties of the conditionally invariant measure and of $\mathcal{L}$ this can be transformed into

$$\langle c_1(f^m)c_2\rangle=\lambda_c^{-m}\int_{\mathcal{U}}c_1(y)\mathcal{L}^m[c_2P](y)dy. \tag{2}$$

The correlation function is obtained by the replacement

$$c_{1,2}\to c_{1,2}-\langle c_{1,2}\rangle.$$

$\langle c_1\rangle$ is given by setting $c_2\equiv 1$ in Eq. (2). Thus we get (we can avoid here specifying the average of c_2)

$$c_{12}(m)=\lambda_c^{-m}\int_{\mathcal{U}}dy(c_1(y)-\langle c_1\rangle)\mathcal{L}^m[c_2P](y), \tag{3}$$

$$\langle c_1\rangle=\int_{\mathcal{U}}c_1(x)P(x)dx.$$

The correlation function defined in this way depends obviously on the neighborhood U. But in spite of the nonunique-

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ness of U this dependency is not particularly relevant since all trajectories move away from the repeller exponentially fast. Therefore the asympotics of all the $c_{12}(m)$ is the same and different neighborhoods will affect only the corrections to these asymptotics, not the asymptotics itself. This is not in contrast to the statement that the correlation function *on* the repeller will be different from $c_{12}(m)$ defined here. (The correlation function *on* the repeller was discussed by Csordás [5].) The former corresponds to taking the limit $U\rightarrow 0$ first and then computing the correlations for large m. In contrast the correlation function c_{12} is computed for arbitrary large m first and afterwards U may be restricted to an arbitrary small neighborhood of the repeller. This is an interchange of two limits and we give an explicit example below for which the result is different. The function defined in Eq. (3) can be called naturally *the correlation function of transient chaos*.

Conditionally invariant measures have been introduced some time ago when discussing 1D maps having a repeller [1]. In this paper we will look into the properties of a class of 1D maps defined implicitly on the interval $[0,1]$:

$$f(x)=f_0(x)-v(f(x))$$

$$v(x)=v(1-x), \quad v(0)=0, \quad -1\leqslant v'(x)\leqslant 1 \tag{4}$$

and

$$f_0(x)=\begin{cases} 2R_g x & \text{for } 0\leqslant x\leqslant 1/2R_g \\ 2R_g(1-x) & \text{for } 1-1/2R_g\leqslant x\leqslant 1. \end{cases} \tag{5}$$

These maps represent a very general class of repellers [7] and at the same time have simple inverse mappings and a simple conditionally invariant measure: The inverse of the lower and upper branch, respectively, are given by

$$f_l^{-1}(x)=\frac{x+v(x)}{2R_g},$$

$$f_u^{-1}(x)=1-f_l(x) \tag{6}$$

and the conditionally invariant measure is obtained by inspection,

$$P(x)=1+v'(x) \tag{7}$$

with the eigenvalue

$$\lambda_c=1/R_g \tag{8}$$

and an escape rate

$$\kappa=\ln R_g . \tag{9}$$

The general case will be discussed in a forthcoming paper. Here we discuss only maps fulfilling the condition

$$v'(0)=-v'(1)=1. \tag{10}$$

This means that at the two end points of the window the slope becomes infinite, cf. Eq. (6). The properties of such maps are very rich. In Sec. II we will prove analytically as well as numerically that for $R_g>1$ a repeller generated by any of these maps always has a natural invariant measure that is a δ function at the origin. (This has been pointed out already in [7].) At $R_g=1$ the system becomes intermittent and nonrepelling characterized by two coexisting invariant measures (one being a δ function, the other being a smooth measure, namely, the former conditionally invariant measure). Thus there exists a first order phase transition at $R_g=1$ for these maps [6]. The appropriate control parameter is

$$\varepsilon=R_g-1. \tag{11}$$

The first eigenfunction of $\mathcal{L}$ with eigenvalue $\lambda_0<\lambda_c$ turns out to be the most important one. Its eigenvalue λ_0 is only by an amount $O(\varepsilon^2)$ smaller than λ_c leading to a very slow decay of correlations and a correlation length $\Lambda\propto\varepsilon^{-2}$ (in this connection *length* means always the number of iterations). This eigenfunction and its eigenvalue are computed in Sec. III. In Sec. IV we approximate the other eigenvalues and eigenfunctions by analytic expressions, compute the correlation function $c_{12}(m)$ analytically, and investigate scaling properties. We compare the analytic expressions with numerical results and show that the agreement is very good indeed. From the analytic formula we recognize in particular the crossover length Λ_{cross} at which the exponent of the power law decay changes from 1 to 0. We find the anomalous ratio $\Lambda_{\text{cross}}/\Lambda\propto\varepsilon$. The conclusion ends the paper.

II. THE NATURAL MEASURE OF THE REPELLER

The structure of the repeller will be determined by introducing the function

$$\theta^{(N)}(x)=\begin{cases} \lambda_c^{-N} & \text{for } 0\leqslant f^N(x)\leqslant 1 \quad (12)\\ 0 \text{ otherwise} & \quad (13)\end{cases}$$

and its limit

$$\theta=\lim_{N\rightarrow\infty}\theta^{(N)}. \tag{14}$$

We obtain for the natural invariant measure of the repeller in the Nth step [7–9]

$$\rho^{(N)}(x)=\theta^{(N)}(x)P(x) \tag{15}$$

and for the invariant measure

$$\rho(x)=\lim_{N\rightarrow\infty}\rho^{(N)}. \tag{16}$$

The computation of $\rho(x)$ can be done numerically for $R_g\gg 1$ by directly applying this iteration scheme [7]. But we are interested in the phase transition $R_g\rightarrow 1$ and in that limit this scheme is prohibitive.

We can use an analytic method beginning with the observation that the natural measure of the repeller is related to the first eigenfunction of the adjoint Frobenius-Perron operator defined as

$$\mathcal{L}^{+}g=g(f(x))F(x),$$
$$F(x)=\begin{cases}1 \text{ for } 0\leqslant x\leqslant 1/2R_g \text{ or } 1-1/2R_g\leqslant x\leqslant 1\\ 0 \text{ otherwise.}\end{cases} \tag{17}$$

The basic eigenfunction of the adjoint operator can be obtained via the construction [we are allowed to start with the function 1 since∫011×P(x)dx≠0)]

$$\tau^{(N)}=\lambda_c^{-N}\mathcal{L}^{+N}1. \tag{18}$$

The first eigenfunction is the limit

$$\tau=\lim_{N\to\infty}\tau^{(N)}. \tag{19}$$

(For $R_g=1$ $\mathcal{L}^{+}1=1$, so in that case 1 is an eigenfunction with eigenvalue $\lambda_c=1$. We will see that this eigenvalue is degenerate.) We realize at once that τ and θ are *identical*. On the other hand, the first eigenfunction can be obtained by inspection. It is

$$\tau(x)=\delta(x)+\delta(1-x), \tag{20}$$

since we can write

$$\delta(f(x))=\frac{1}{|f'(0)|}\delta(x)+\frac{1}{|f'(0)|}\delta(1-x)$$

and because of Eqs. (10) and (6)

$$\delta(1-f(x))=\frac{1}{|f'(f^{-1}(1))|}\delta(f^{-1}(1)-x)=0.$$

Therefore, using Eqs. (7), (10), (15), and (16), the density of the natural measure turns out to be a δ function at the origin

$$\rho(x)=\delta(x) \quad \text{for} \quad R_g>1. \tag{21}$$

It was proven previously [7] that the natural measure possesses a δ function contribution at the origin and it was suggested — based on numerical results — that the prefactor would be 1.

Approaching $R_g=1$ on a different route, namely, that of fully developed chaos [10,11], the density of the natural measure is P. Besides the smooth density P there exists at the phase transition point the δ function, which is being an eigenfunction of the Frobenius-Perron operator with the same eigenvalue $\lambda_c=1$. The coexistence of these two measures shows that the phase transition is of first order.

III. THE EIGENFUNCTION WITH SECOND LARGEST EIGENVALUE

Having the first eigenfunction of $\mathcal{L}^{+}$ we can compute the first eigenfunction of $\mathcal{L}$ with eigenvalue $\lambda_0<\lambda_c$ in the following manner: (i) start with an arbitrary function $\psi^{(0)}$; (ii) compute

$$p(x)=\mathcal{L}\psi^{(0)};$$

(iii) project the contribution of $P(x)$ out and normalize

$$\psi^{(1)}(x)=\frac{p(x)-\langle\tau|p\rangle P(x)}{\|p-\langle\tau|p\rangle P\|};$$

(iv) iterate this procedure. As a result, the first nontrivial eigenfunction ψ_0 and its first eigenvalue λ_0 are obtained fulfilling

$$\mathcal{L}\psi_0=\lambda_0\psi_0 \tag{22}$$

In our case we have

$$\langle\tau|p\rangle=p(0) \tag{23}$$

and inserting this we find a *numerical* solution ψ_0 that fulfills the eigenvalue equation with a numerical error $<10^{-10}$. This holds true at least in the range $1.000\,01\leqslant R_g\leqslant 10$. [The small error also provides numerical evidence that Eq. (23) is correct and that the measure of the repeller is indeed a δ function.] Numerical results for the eigenfunctions are given in Fig. 1. Here and throughout the paper the numerics were done inserting

$$v(x)=x(1-x). \tag{24}$$

In Fig. 2 $\lambda_0(R_g)$ is shown. We observe that $P(x)$ and $\psi_0(x)$ are nearly degenerate for $\varepsilon\to 0$, which lets us expect that the first nontrivial eigenfunction is particularly important and that, e.g., the correlation length Λ is determined by the properties of the first nontrivial eigenvalue. Therefore analytic approximations are desirable and will be given next.

Because of the δ function character of the natural measure we expect the eigenfunctions $\psi_n(x)$ to be most important for small arguments. But for small arguments the second branch of the Frobenius-Perron operator being $\propto\psi_n[1-f'(0)x]$ can be neglected [10]:

$$\psi_n(1-y)=\lambda_n^{-1}\frac{1}{|f'(f_l^{-1}(1-y))|}[\psi_n(f_l^{-1}(1-y)) + \psi_n(f_u^{-1}(1-y))]$$

but

$$\frac{1}{|f'(f_l^{-1}(1))|}=\frac{1}{|f'(f_u^{-1}(1))|}=0$$

and therefore

$$\lim_{y\to 0}\psi_n(1-y)=0. \tag{25}$$

Then the eigenvalue equation can be approximated by a differential equation [10] and with the definition

$$\beta=\frac{2\varepsilon}{f''(0)} \tag{26}$$

one obtains the approximate solution

$$\lambda_0^{(0)}=\lambda_c+O(\varepsilon^2), \tag{27}$$

$$\psi_0^{(0)}(x)=\frac{\beta^2}{(x+\beta)^2}-\frac{1}{2}P(x). \tag{28}$$

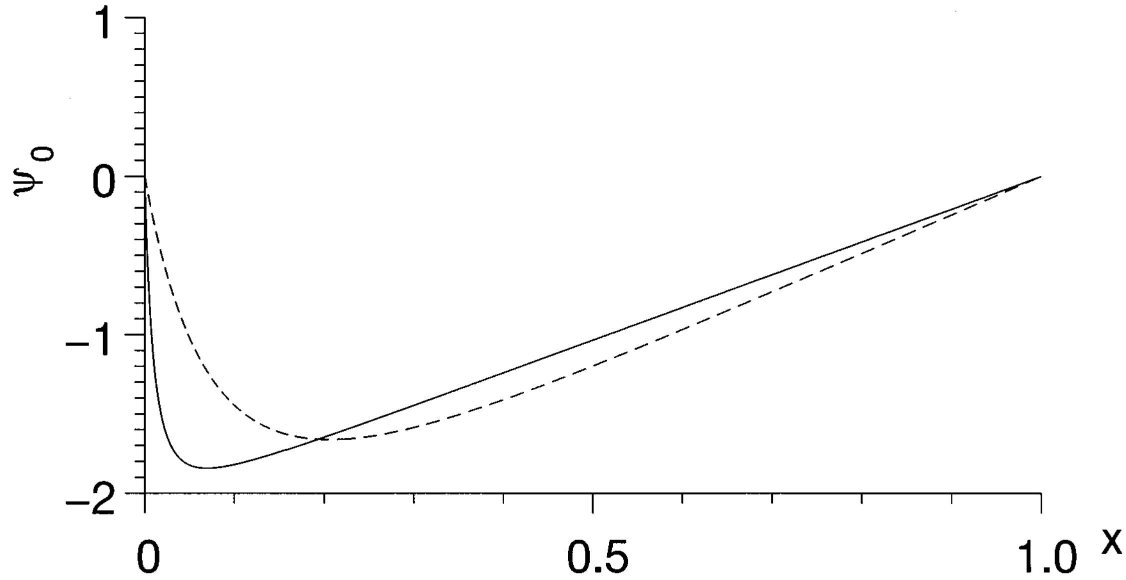

FIG. 1. Eigenfunction $\psi_0(x)$ with second largest eigenvalue λ_0 of the Frobenius-Perron operator. The eigenfunction is normalized, i.e., $\int_0^1 \psi_0 = 1$. Solid line: $\varepsilon = 0.01$, dashed line: $\varepsilon = 0.1$.

[The second term has been added to fulfill $\psi_0^{(0)}(0) = 0$.] Whereas this is quite a good approximation for the eigenfunction as long as ε is small (cf. Fig. 3) the approximation for the eigenvalue is too crude. To get a better approximation we iterate

$$\psi_0^{(1)} = 2R_g \mathcal{L} \psi_0^{(0)} - \text{const} \times P. \tag{29}$$

const is determined from the condition $\psi_0^{(1)}(0) = 0$. This leads to

$$\psi_0^{(1)}(x) = P(x)\left(\frac{\beta^2}{[f_l^{-1}(x)+\beta]^2} + \frac{\beta^2}{[f_u^{-1}(x)+\beta]^2} - \sigma \right), \tag{30}$$

$$\sigma = 1 + \frac{\beta^2}{(1+\beta)^2}.$$

We expect $\psi_0^{(1)}$ to be a very good approximation since it fulfills Eq. (25) as well and thus

$$\psi_0 = \lim_{N \to \infty} \frac{1}{\lambda_0^N} \mathcal{L}^N \psi_0^{(1)}. \tag{31}$$

A comparison between ψ_0 and $\psi_0^{(1)}$ is shown in Fig. 4 and the agreement is very good indeed. We compute the eigenvalue $\lambda_0^{(1)}$ from

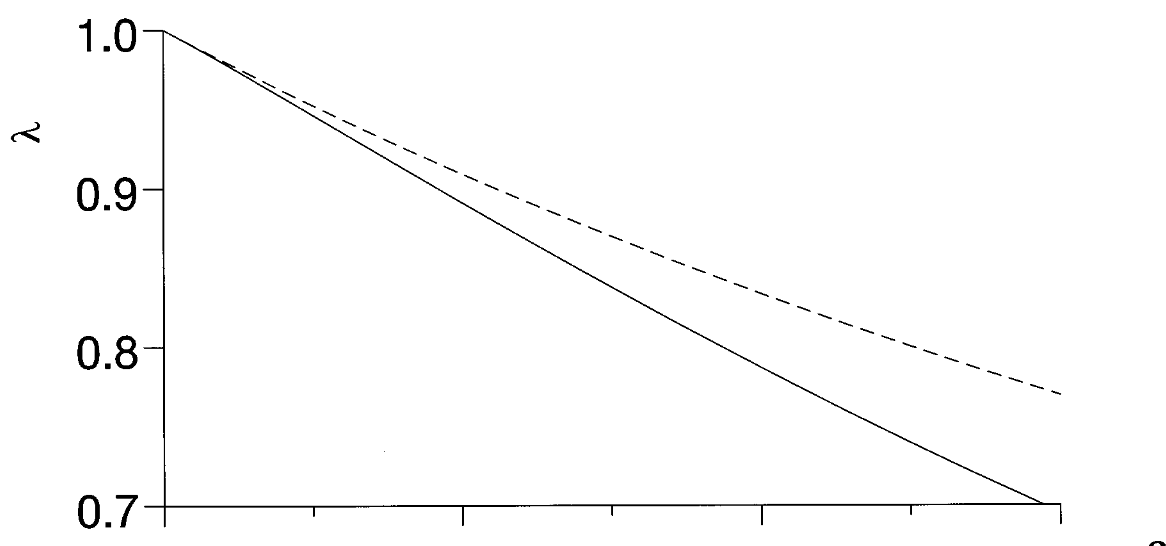

FIG. 2. Eigenvalue of the conditionally invariant measure, λ_c (solid line), and the second largest eigenvalue λ_0 (dashed line), as function of ε. They merge for $\varepsilon \to 0$.

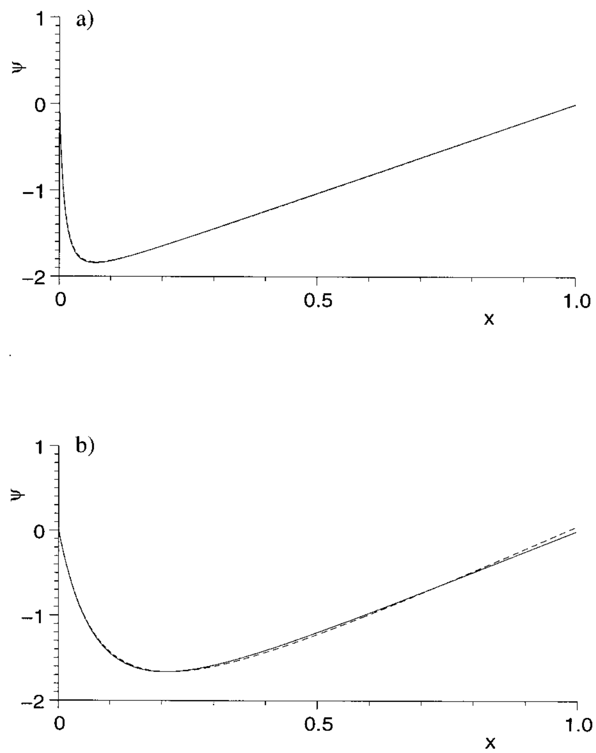

FIG. 3. Comparison between the exact eigenfunction $\psi_0(x)$ (solid line) and $\psi_0^{(0)}(x)$, the simplest approximation in this paper (dashed line). The eigenfunctions are normalized. (a) $\varepsilon=0.01$, (b) $\varepsilon=0.1$.

$$\lambda_0^{(1)}=\frac{\int_0^1 \mathcal{L}\psi_0^{(1)}}{\int_0^1 \psi_0^{(1)}} \tag{32}$$

The computation is done in the Appendix. The approximation used here is compared with exact numerical results; cf. Fig. 5. The agreement between λ_0 and $\lambda_0^{(1)}$ is again very good. $\lambda_0(R_g)$ depends on v; cf. Eq. (4). However, the leading term in an ε expansion *is universal*:

$$\lambda_0=\lambda_c[1-2\varepsilon\beta]+O(\varepsilon^3). \tag{33}$$

There is no degeneracy for *finite* ε but a near degeneracy with the result that initial distributions steeply peaked at zero will decay very slowly and the correlation length is

$$\Lambda \propto \varepsilon^{-2}. \tag{34}$$

IV. THE CORRELATION FUNCTION

To compute the correlation function we need not only ψ_0 but all the other eigenfunctions of the Frobenius-Perron operator as well. Proceeding as in an earlier paper [11], we get for the approximate eigenfunctions

$$\phi_n^{(0)}(x)=\frac{\beta(1+\beta)^{n+1}x^n}{(x+\beta)^{n+2}},$$

$$\lambda_n^{(0)}=\lambda_0 e^{-\varepsilon n}. \tag{35}$$

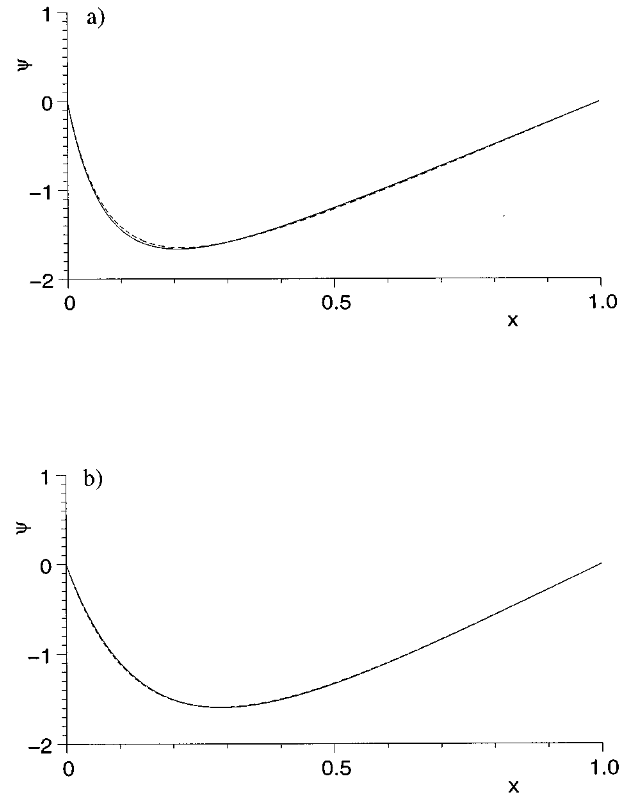

FIG. 4. Comparison between the exact eigenfunction $\psi_0(x)$ (solid line) and the approximate one $\psi_0^{(1)}(x)$ (dashed line). The eigenfunctions are normalized. (a) $\varepsilon=0.1$, (b) $\varepsilon=0.3$.

This approximation is sufficient for $n>0$ since $\phi_n(0)=0$ is fulfilled and Eq. (25) is nearly fulfilled for small ε. For large ε the higher eigenfunctions are not very relevant for the computation of the correlation function anyway.

We assume here that the c_i are analytic in $[0,1]$. Expanding $[c_2-\langle c_2\rangle]P$ in the series of the ϕ_n we may write

$$\mathcal{L}^m[c_2-\langle c_2\rangle]P=\mathcal{L}^m\sum_{\nu=0}^{\infty} a_\nu\phi_\nu$$

$$\approx \lambda_0^m\sum_{\nu=0}^{\infty} a_\nu e^{-\varepsilon m\nu}\phi_\nu+a_0(\mathcal{L}^m-\lambda_0^m)\phi_0.$$

The conjugation

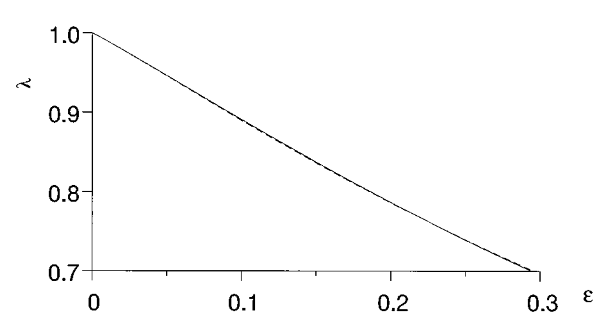

FIG. 5. Comparison between the exact eigenvalue λ_0 (solid line) and the approximate one $\lambda_0^{(1)}$ (dashed line).

$$y=h(x)=\frac{x}{x+\beta}[1+\beta] \tag{36}$$

transforms the ϕ_n just into powers. Therefore the the conjugation of Σ is a Taylor series and we obtain a closed expression for it. Furthermore a_0 is given by

$$a_0=[c_2(0)-\langle c_2\rangle]P(0)\beta(1+\beta) \tag{37}$$

and we can use the fact that $\psi_0^{(1)}$ is a good approximation for the first nontrivial eigenfunction. Putting all this together we find

$$\mathcal{L}^m d_2 \approx \lambda_0^m \frac{1}{[(1-\lambda_0^m)(a/\varepsilon)x+1]^2} \times d_2\left[\lambda_0^m\left(\frac{x}{(1-\lambda_0^m)(a/\varepsilon)x+1}\right)\right] + c_2(0)\lambda_0^m\left[\frac{\lambda_c}{\lambda_0}\psi_0^{(1)}-\psi_0^{(0)}\right]+\lambda_c^m\sigma P. \tag{38}$$

Let us assume from now on that $[c_1(0)-\langle c_1\rangle]\neq 0$ and $[c_2(0)-\langle c_2\rangle]\neq 0$ (this is the generic case). Note that any term $\propto P(x)$ in Eq. (38) is negligible since $\int_{\mathcal{U}}[c_1-\langle c_1\rangle]P=0$. Furthermore it turns out that the second term of Eq. (38) can be neglected [this term gives a relative contribution $O(\varepsilon^2)$]. We take for $\mathcal{U}$ the whole interval [0,1] deferring the discussion of smaller $\mathcal{U}$ to the end of the section. The result is

$$c_{12}(m)\approx\frac{(\lambda_0/\lambda_c)^m}{(1-\lambda_0^m)a/\varepsilon}[c_1(0)-\langle c_1\rangle] \times[c_2(0)-\langle c_2\rangle]P(0)\int_0^{(1-\lambda_0^m)\beta^{-1}}\frac{1}{(1+y)^2}. \tag{39}$$

We have checked this formula numerically by using Eq. (3) and setting for c_i

$$c_1(x)=c_2(x)=\begin{cases}1 \text{ if } x<B\\ 0 \text{ otherwise.}\end{cases} \tag{40}$$

(We choose for c_2 a step function because of its numerical advantages. Of course the step function is not analytic. However, cutting the tail of its Fourier series we can construct an analytic function being arbitrarily close to a step function.) In Fig. 6 the ratio $c_{12}^{\text{numeric}}/c_{12}^{\text{analytic}}$ is shown for various ε values. [For the analytic computation of the correlation function we use the asymptotic formula taking into account the finite integration limit; i.e., we compute $\int_0^{B(1-\lambda_0^m)\beta^{-1}} 1/(1+y)^2$.] One observes from the figure that $c_{12}^{\text{numeric}}/c_{12}^{\text{analytic}}$ becomes const but the constant is less than 1. This has a simple explanation: the approximate eigenfunctions have been determined with high accuracy near 0 where they are peaked and consequently the expansion coefficients of c_2 are sufficiently accurate only if the function c_2 is strongly concentrated around 0. If this is not fulfilled one expects deviations. These are not relevant concerning the independence of the correlation function since all the tails of c_2 decay exponentially fast under application of the Frobenius Perron operator. Thus with increasing i $\mathcal{L}^i c_2$ becomes strongly concentrated at 0 and its expansion into the approximate eigenfucntion becomes very accurate but some weight has been lost. As a result one gets the forementioned effect.

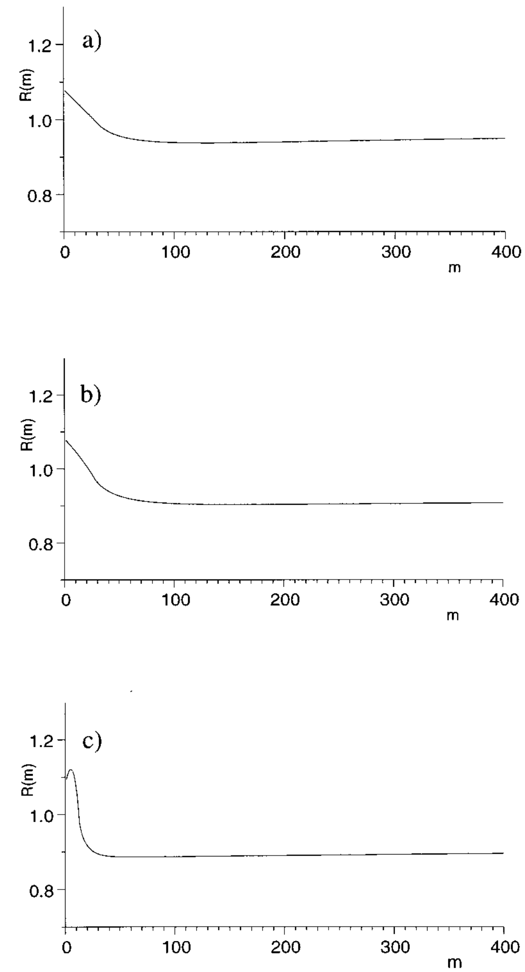

FIG. 6. Comparison between the numerically calculated and the analytically derived correlation function $c_{12}(m)$. Shown is the ratio $R(m)=c_{12}^{\text{numeric}}/c_{12}^{\text{analytic}}$ as a function of m. Note that there is no fit parameter. $B=0.05$; cf. Eq. (40). (a) $\varepsilon=0.001$, (b) $\varepsilon=0.01$, (c) $\varepsilon=0.1$.

We get the following asymptotic behavior:

$$c_{12}(m)\to\text{const}\times\frac{1}{m}(\lambda_0/\lambda_c)^m\frac{\varepsilon m}{1-e^{-\varepsilon m}}. \tag{41}$$

Let us analyze this result a little bit. λ_0/λ_c is $O(e^{-\varepsilon^2})$; cf. Eqs. (26) and (33). Therefore we can separate the m values into three regions; (i) $m\ll\varepsilon^{-1}$: Here we have simple power law decay $\propto m^{-1}$. This region includes $\varepsilon=0$ and the result is in agreement with [11]. (ii) $\varepsilon^{-1}\ll m\ll\varepsilon^{-2}$: In this region the correlation function remains approximately constant. The crossover occurs at $O(m)=\varepsilon^{-1}$, giving the crossover correlation length

$$\Lambda_{\text{cross}}\propto\varepsilon^{-1}. \tag{42}$$

(iii) $\varepsilon^{-2} \ll m$: In this region we find exponential decay $\propto e^{-m\varepsilon^2}$ defining the correlation length as

$$\Lambda \propto \varepsilon^{-2}. \tag{43}$$

Obviously the system has two scales, Λ_{cross} and Λ, in contrast to the situation of fully developed chaos [11]. Furthermore the ratio $\Lambda/\Lambda_{\text{cross}} \propto \varepsilon^{-1}$ and diverges for $\varepsilon \to 0$. Therefore the asymptotics of the correlation function can be written in the form $m^{-1} \times S(\Lambda_{\text{cross}}^{-1} m, \Lambda^{-1} m)$. (See Ref. [12] for an analogous situation in dynamical critical phenomena).

The scaling depends crucially on the values of the $c_i(0)$. This is not surprising since the natural measure of the repeller is a δ function at the origin. Trajectories staying for a transient time m in the neighborhood of the repeller must remain close to 0 if m is large.

Up to now we have taken the interval $[0,1]$ as the neighborhood of the repeller. Now we estimate the corrections when choosing a smaller neighborhood $[0,u_l]$. From Eq. (39) we recognize that the integral will change by $O(1/mu_l)$ for $m < \Lambda_{\text{cross}}$ and by $O(\varepsilon u_l)$ for $m > \Lambda_{\text{cross}}$. These are small corrections not affecting the asymptotics as long as u_l remains finite.

To obtain the correlation function *on* the repeller one has to take the limit $u_l \to 0$ first and then a completely different result is found. [In fact we get 0 because $c_1(x) - \langle c_1 \rangle = 0$ on the natural measure of the repeller.]

V. CONCLUSION

We have derived correlation functions $c_{12}(m)$ for the transients of a repeller and computed them analytically as well as numerically for a particular class of 1D maps. This class is generated by a tent map having a window $\varepsilon/(1+\varepsilon)$, cf. Eqs. (4) and (5) and the slope of these maps is ∞ at the two end points of the window. The intrinsic properties of this class are as follows. (i) A first order phase transition at $\varepsilon = 0$ from a repeller to a nonrepelling intermittent state. (ii) A natural measure, which is a δ function at the origin for arbitrary $\varepsilon > 0$. (iii) For small ε the difference between the leading eigenvalue λ_c and the next one, λ_0, is of order ε^2. This results in a correlation length $\Lambda \propto \varepsilon^{-2}$. (iv) Correlation functions were computed analytically and numerically. They decay with a power law $\propto m^{-1}$ below the crossover length $\Lambda_{\text{cross}} \propto \varepsilon^{-1}$, remain approximately constant in the range $\Lambda_{\text{cross}} < m < \Lambda$. Beyond that they decay exponentially but very slowly because of the large correlation length Λ. We remind the reader that correlation functions of transient chaos are to be defined in the neighborhood of the repeller not on the repeller itself. These neighborhoods cannot be defined uniquely. However, the asymptotics for large m is independent of their definition.

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APPENDIX

We compute the expression

$$\lambda_0^{(1)} = \frac{\int_0^1 \mathcal{L} \psi_0^{(1)}}{\int_0^1 \psi_0^{(1)}}. \tag{A1}$$

Because of Eq. (29) this is

$$\lambda_0^{(1)} = \lambda_c \frac{1 - 2(R_g^2/\sigma) \int_0^1 \mathcal{L}^2 \phi_0}{1 - 2(R_g/\sigma) \int_0^1 \mathcal{L} \phi_0}, \tag{A2}$$

$$\int_0^1 \mathcal{L} \phi_0 = \sum_{i=1}^{2} -\frac{1}{l_{2i}+\beta} + \frac{1}{l_{2i-1}+\beta},$$

$$l_1 = f_l^{-1}(0) = 0,$$

$$l_2 = f_l^{-1}(1) = \frac{1}{2R_g},$$

$$l_3 = 1 - l_2,$$

$$l_4 = 1,$$

$$\int_0^1 \mathcal{L}\mathcal{L} \phi_0 = \sum_{i=1}^{4} -\frac{1}{l_{2i}+\beta} + \frac{1}{l_{2i-1}+\beta}$$

$$l_1 = f_l^{-1}(0) = 0,$$

$$l_2 = f_l^{-1}\left(\frac{1}{2R_g}\right) = \frac{1}{4R_g^2}\left[1 + 2R_g v\left(\frac{1}{2R_g}\right)\right],$$

$$l_3 = f_l^{-1}\left(1 - \frac{1}{2R_g}\right) = \frac{1}{2R_g} - \frac{1}{4R_g^2}\left[1 - 2R_g v\left(\frac{1}{2R_g}\right)\right],$$

$$l_4 = f_l^{-1}(1) = \frac{1}{2R_g},$$

$$l_5 = 1 - l_4,$$

$$l_6 = 1 - l_3,$$

$$l_7 = 1 - l_2,$$

$$l_8 = 1.$$

Because of Eq. (31) this scheme can easily be extended to arbitrary high order and has been used for precise numerical computations of λ_0.

For small ε we get an exact *universal* expansion up to ε^2. To achieve that let us normalize the true eigenfunction ψ_0 such that

$$\int_0^1 \psi_0 = \int_0^1 \psi_0^{(1)}. \tag{A3}$$

Let us assume furthermore

$$\psi_0^{(1)}(x) - \psi_0(x) = O(\varepsilon^2) \quad \text{if} \quad O(x) = 1. \tag{A4}$$

[This assumption is very plausible since the eigenvalue equation is fulfilled by $\psi_0^{(1)}$ up to $O(\varepsilon^2)$.] Then we find

$$\int_0^1 \mathcal{L}\psi_0 = \int_0^1 \psi_0 - \int_{1/2R_g}^{1-1/2R_g} \psi_0$$
$$= \int_0^1 \psi_0^{(1)} - \int_{1/2R_g}^{1-1/2R_g} \psi_0^{(1)} + O(\varepsilon^3)$$
$$= \int_0^1 \mathcal{L}\psi_0^{(1)} + O(\varepsilon^3) = 2R_g \int_0^1 \phi_0 - \lambda_c \sigma + O(\varepsilon^3)$$

and thus

$$\frac{\lambda_0}{\lambda_c} = \frac{1 - 2(R_g^2/\sigma)\int_0^1 \phi_0}{1 - 2(R_g/\sigma)\int_0^1 \phi_0} + O(\varepsilon^3)$$

or

$$\frac{\lambda_0}{\lambda_c} = 1 - 2\varepsilon\beta + O(\varepsilon^3). \tag{A5}$$

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